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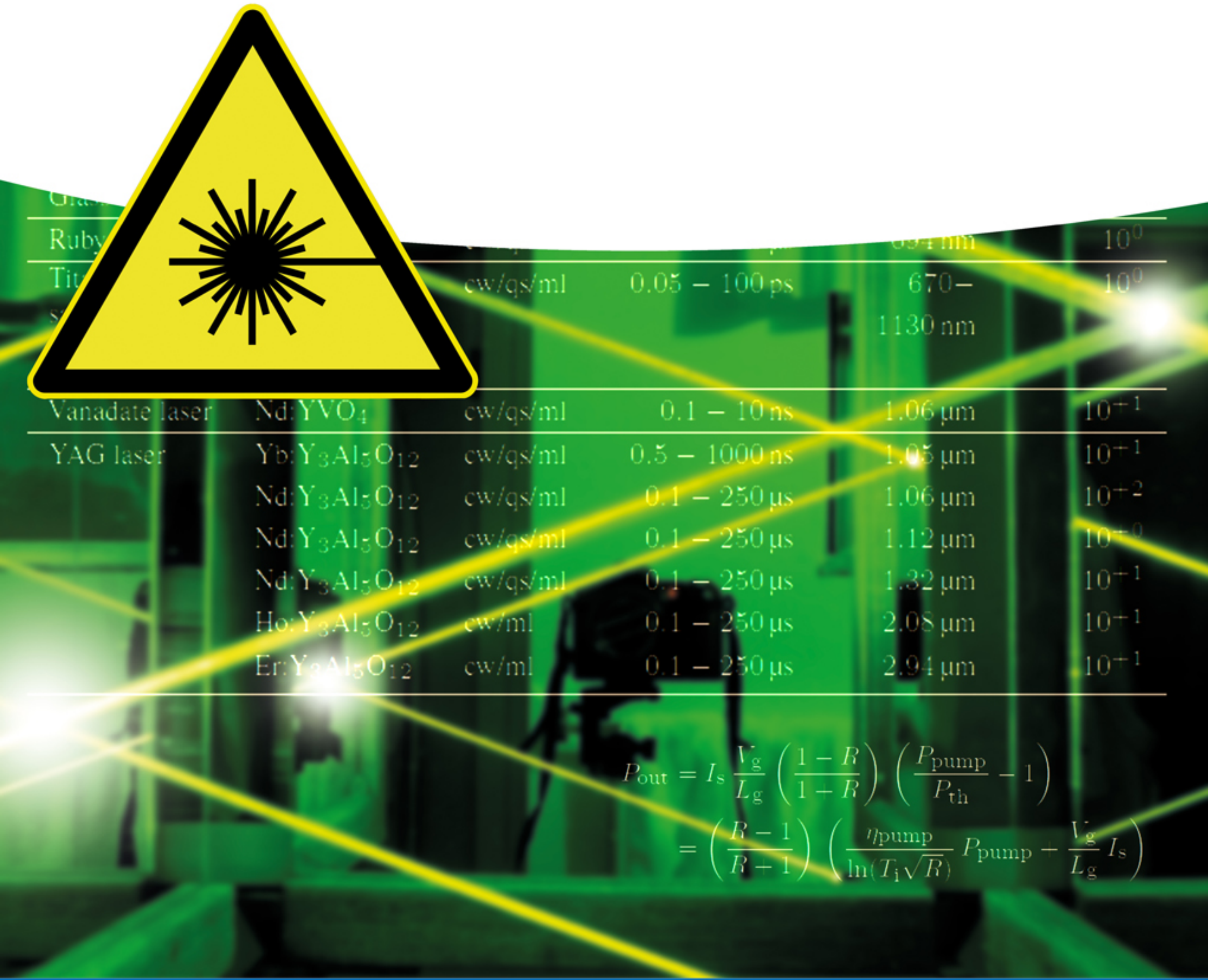
Basics of Laser Systems



Solutions to Problems in „Optical Devices in Ophthalmology and Optometry“

Contents

PB.1
Einstein relations

Starting from Eq. (B5), Einstein found that the probabilities for absorption and stimulated emission are equal for a two-level system. So, he supposed that the number of atoms in the ground state and in the excited state (N_1 and N_2, respectively) are given by the Boltzmann distribution

$$\frac{N_2}{N_1} = \exp\left(-\frac{\hbar\omega_{21}}{k_\mathrm{B}T}\right) \quad ,$$

where k_B is the Boltzmann constant. In addition, he used Planck's formula for the energy density

$$\rho(\omega) = \frac{\hbar\omega^3}{\pi^2 c^3}\left(\exp\left(\frac{\hbar\omega}{k_\mathrm{B}T}\right) - 1\right)^{-1} \quad .$$

Follow Einstein's approach and show that Eq. (B6) can be derived from Eq. (B5).

Solution:

We start from Eq. (B5) which reads

$$B_{12}N_1\,\rho(\omega) = A_{21}N_2 + B_{21}N_2\,\rho(\omega) \quad .$$

Further we have

$$\frac{N_2}{N_1} = \exp\left(-\frac{\hbar\omega_{21}}{k_\mathrm{B}T}\right) \quad . \tag{SB.1}$$

Because of

$$\rho(\omega) = \frac{\hbar\,\omega^3}{\pi^2\,c^3}\frac{1}{\exp\left(\frac{\hbar\omega_{21}}{k_\mathrm{B}T}\right) - 1} \tag{SB.2}$$

we obtain

$$B_{12}N_1\left(\frac{\hbar\,\omega^3}{\pi^2\,c^3}\right)\left(\frac{1}{N_1/N_2 - 1}\right) = A_{21}N_2 + B_{21}N_2\left(\frac{\hbar\,\omega^3}{\pi^2\,c^3}\right)\left(\frac{1}{N_1/N_2 - 1}\right) \quad . \tag{SB.3}$$

We re-arrange Eq. (SB.3) using Eqs. (SB.1) and (SB.2) so that

$$B_{12}\left(\frac{\hbar\omega^3}{\pi^2c^3}\right)\left(\frac{N_1N_2}{N_1 - N_2}\right) = A_{21}N_2 + B_{21}\left(\frac{\hbar\omega^3}{\pi^2c^3}\right)\left(\frac{N_2^2}{N_1 - N_2}\right) \quad ,$$

$$B_{12}\left(\frac{\hbar\omega^3}{\pi^2c^3}\right)N_1N_2 = A_{21}N_1N_2 - A_{21}N_2^2 + B_{21}\left(\frac{\hbar\omega^3}{\pi^2c^3}\right)N_2^2 \quad ,$$

$$B_{12}\left(\frac{\hbar\omega^3}{\pi^2c^3}\right)N_2^2\cdot\exp\left(\frac{\hbar\omega_{21}}{k_\mathrm{B}T}\right) = A_{21}N_2^2\cdot\exp\left(\frac{\hbar\omega_{21}}{k_\mathrm{B}T}\right) - A_{21}N_2^2 + B_{21}\left(\frac{\hbar\omega^3}{\pi^2c^3}\right)N_2^2 \quad .$$

Division by N_2^2 finally leads to

$$\left(B_{12}\left(\frac{\hbar\omega^3}{\pi^2 c^3}\right) - A_{21}\right) \cdot \exp\left(\frac{\hbar\omega_{21}}{k_{\mathrm{B}}T}\right) = B_{21}\left(\frac{\hbar\omega^3}{\pi^2 c^3}\right) - A_{21} \quad . \tag{SB.4}$$

Irrespective of the temperature T, Eq. (SB.4) must be fulfilled, which can only be achieved if

$$B_{12}\left(\frac{\hbar\omega^3}{\pi^2 c^3}\right) = A_{21} \text{ and}$$
$$B_{12} = B_{21} \quad .$$

PB.2
Stability condition

A paraxial light beam bouncing forth and back in a resonator can be considered as if the beam would pass through a periodic sequence of lenses. Each component is described by an ABCD matrix (Section A.1.3).

1. To derive the general stability condition (B33) for resonators, we calculate

$$\begin{pmatrix} h_2 \\ \gamma_2 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} h_0 \\ \gamma_0 \end{pmatrix} \quad . \tag{B.42}$$

   Solve the resulting quadratic equation by using the ansatz

$$h_m = h_0 K^m \tag{B.43}$$

   with $K = \text{const}$ and $m = 0, 1, 2, \ldots$. In the case of lens systems, $\det \underline{\mathbf{M}} = AD - BC = 1$. A periodical and stable solution for Eq. (B43) is obtained if the linear combination $K = K_+ - K_-$ is a real number.
2. Let us now consider the special case of a Gaussian resonator. The corresponding ABCD matrix can be calculated from the matrices of a thin lens and free space according to

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2/r_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & L_{\mathrm{R}} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2/r_2 & 1 \end{pmatrix} \begin{pmatrix} 1 & L_{\mathrm{R}} \\ 0 & 1 \end{pmatrix} \quad . \tag{B.44}$$

   Derive the condition (B36) from Eq. (B33) with the dimension parameters given in Eqs. (B34) and (B35).

Solution:

1. We know from Section A.1.3 that

$$\begin{aligned}\underline{\mathbf{M}}^n &= \begin{pmatrix} A & B \\ C & D \end{pmatrix}^n \\ &= \frac{1}{\sin\theta}\begin{pmatrix} A\ \sin n\theta - \sin(n-1)\theta & B\ \sin n\theta \\ C\ \sin n\theta & D\ \sin n\theta - \sin(n-1)\theta \end{pmatrix} \ .\end{aligned}$$

In our case, we have $n = 2$ and det $\underline{\mathbf{M}}^2 = (AD - BC) = 1$, which leads to

$$\begin{aligned}\underline{\mathbf{M}}^2 &= \begin{pmatrix} A & B \\ C & D \end{pmatrix}^2 \\ &= \frac{1}{\sin\theta}\begin{pmatrix} A\ \sin 2\theta - \sin\theta & B\ \sin 2\theta \\ C\ \sin 2\theta & D\ \sin 2\theta - \sin\theta \end{pmatrix} \ .\end{aligned}$$

When using the trigonometric identity $\sin 2\theta = 2\ \sin\theta\cos\theta$, it follows that

$$\begin{aligned}\underline{\mathbf{M}}^2 &= \begin{pmatrix} A & B \\ C & D \end{pmatrix}^2 \\ &= \begin{pmatrix} 2A\ \cos\theta - 1 & 2B\ \cos\theta \\ 2C\ \cos 2\theta & 2D\ \cos\theta - 1 \end{pmatrix} \ .\end{aligned}$$

Because of

$$\det\ \underline{\mathbf{M}}^2 = (2A\ \cos\theta - 1)(2D\ \cos\theta - 1) - (2B\ \cos\theta)(2C\ \cos\theta) = 1 \ ,$$

we obtain the equations

$$\begin{aligned}&4AD\ \cos^2\theta - 2A\ \cos\theta - 2D\ \cos\theta + 1 - 4BC\ \cos^2\theta = 1 \ , \\ &4AD\ \cos\theta - 2A - 2D - 4BC\ \cos\theta = 0 \ , \\ &(AD - BC)\ \cos\theta = \frac{A + D}{2} \ ,\end{aligned}$$

and finally

$$\cos\theta = \frac{A + D}{2} \ .$$

Since $|\cos\theta| \leq 1$ and $|\frac{A+D}{2}| \leq 1$, we end up with the stability condition

$$-1 \leq \frac{A + D}{2} \leq +1 \ .$$

2. We first calculate the matrix products, that is,

$$\begin{aligned}\begin{pmatrix} A & B \\ C & D \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ 2/r_1 & 1 \end{pmatrix}\begin{pmatrix} 1 & L_{\mathrm{R}} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 2/r_2 & 1 \end{pmatrix}\begin{pmatrix} 1 & L_{\mathrm{R}} \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & L_{\mathrm{R}} \\ 2/r_1 & \frac{2L_{\mathrm{R}}}{r_1} + 1 \end{pmatrix}\begin{pmatrix} 1 & L_{\mathrm{R}} \\ 2/r_2 & \frac{2L_{\mathrm{R}}}{r_1} + 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 + \frac{2L_{\mathrm{R}}}{r_2} & 2L_{\mathrm{R}} + \frac{2L_{\mathrm{R}}^2}{r_2} \\ \frac{2}{r_1} + \frac{2}{r_2} + \frac{4L_{\mathrm{R}}}{r_1 r_2} & \frac{4L_{\mathrm{R}}}{r_1} + \frac{4L_{\mathrm{R}}^2}{r_1 r_2} + \frac{2L_{\mathrm{R}}}{r_2} + 1 \end{pmatrix} \ . \qquad \text{(SB.5)}\end{aligned}$$

In Eqs. (B34) and (B35) we have defined

$$g_1 = 1 + \frac{L_\mathrm{R}}{r_1} \ ,$$
$$g_2 = 1 + \frac{L_\mathrm{R}}{r_2}$$

which can be re-written as

$$\frac{L_\mathrm{R}}{r_1} = g_1 - 1 \ ,$$
$$\frac{L_\mathrm{R}}{r_2} = g_2 - 1 \ .$$

With reference to Eq. (SB.5), we express the matrix elements as

$$\begin{aligned} A &= 1 + \frac{2L_\mathrm{R}}{r_2} \\ &= 1 + 2(g_2 - 1) \\ &= 2g_2 - 1 \ , \\ D &= \frac{4L_\mathrm{R}}{r_1} + \frac{4L_\mathrm{R}^2}{r_1 r_2} + \frac{2L_\mathrm{R}}{r_2} + 1 \\ &= 4g_1 g_2 - 2g_2 - 1 \ . \end{aligned}$$

From this, we obtain

$$\frac{A+D}{2} = \frac{(2g_2 - 1) + (4g_1 g_2 - 2g_2 - 1)}{2} = 2g_1 g_2 - 1 \ .$$

According to Eq. (B33), we know that

$$-1 \leq \frac{A+D}{2} \leq +1 \ .$$

Therefore, we conclude that

$$0 \leq g_1 g_2 \leq 1 \ .$$

PB.3
Gaussian laser beams

We consider a helium–neon laser beam in a TEM_{00} mode which has a waist radius of $w_0 = 1.3\,\text{mm}$. The beam shall be expanded and subsequently focused by an optical system.

1. Calculate the ABCD matrix of an optical system which consists of a negative and a positive lens which have a distance of L. What is the condition for an afocal Galilei telescope? How does the matrix change in this case?
2. The laser beam shall be expanded to a diameter of $2w_0 = 8\,\text{mm}$ by using a Gailiei telescope which consists of thin lenses with a face-to-face length of $L = 50\,\text{mm}$. Calculate the focal lengths of the lenses in the Galilei system.
3. Next, the collimated expanded beam shall be focused so that we obtain a depth of field of $\Delta z = 1\,\text{mm}$. Here, the depth of field is defined as the range at which the beam intensity does not fall below 80% of the maximum intensity at the waist. What is the minimum focal length to achieve this? How large is the diameter of the focus?
4. We assume that the focusing lens, with a minimum focal length as calculate in 3., is placed along the laser path such that the waist (diameter of $8\,\text{mm}$) lies $300\,\text{mm}$ in front of the lens. Calculate the waist position behind the lens relative to the image-side focal point F' of the lens. Does the waist lie in front of or behind the focal point F'?
5. Consider the change of the Gaussian beam parameters when the beam passes through an afocal Kepler- and Galilei-type telescope system. Such an optical system can be used to expand or compress the beam diameter. Calculate the minimum beam diameter and the divergence angle. How are these parameters related to each other? Consider also the product of minimum beam diameter and divergence angle.

Solution:

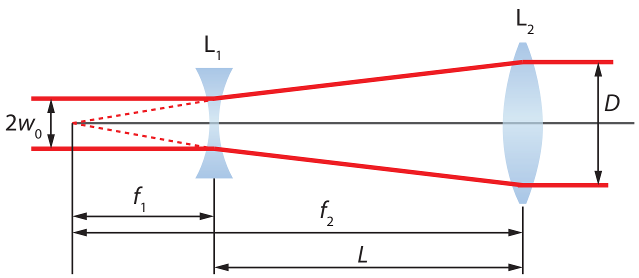


Figure SB.1 Ray diagram of a Galilei telescope.

1. The matrix for a Galilei telescope follows as (Figure SB.1, Table A.1)

$$\begin{aligned}\begin{pmatrix} A & B \\ C & D \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ -1/f_2 & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1/f_1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 - L/f_1 & L \\ -1/f_1 - 1/f_2 + L/(f_1 f_2) & 1 - L/f_2 \end{pmatrix} \ ,\end{aligned}$$

where f_1 is the focal length of the positive lens and f_2 the focal length of the negative lens. The system is afocal if for $\gamma = 0$, γ' is always zero for all h. From the ansatz

$$\begin{pmatrix} h' \\ \gamma' \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot \begin{pmatrix} h \\ \gamma \end{pmatrix} \ .$$

follows that

$$\gamma' = C \cdot h + D \cdot \gamma \ .$$

Thus, we have an afocal system if $C = 0$. In the case of an afocal Galilei system, this is equivalent to

$$\begin{aligned}-1/f_1 - 1/f_2 + L/(f_1 f_2) = 0 \\ \Rightarrow L = f_1 + f_2 \ .\end{aligned}$$

The matrix for the afocal Galilei telescope then reads

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} -f_2/f_1 & f_1 + f_2 \\ 0 & -f_1/f_2 \end{pmatrix} \ .$$

2. The laser beam shall be now expanded to a diameter of $2w_0' = 8$ mm by using a Galilei telescope which consists of thin lenses with a face-to-face length of $L = 50$ mm. The expansion factor of the diameter is

$$\frac{2w_0'}{2w_0} = 3.08 \ .$$

This is equivalent to the reciprocal value of the angular magnification

$$3.08 = \frac{1}{\Gamma} = -\frac{f_2}{f_1} \ .$$

Here, we also used (because of $\gamma = 0$)

$$\begin{aligned} h' &= -\frac{f_2}{f_1}\, h + (f_1 + f_2)\, \gamma = -\frac{f_2}{f_1}\, h \ , \\ \frac{1}{\Gamma} &= -\frac{f_2}{f_1} = \frac{2w_0'}{2w_0} = 3.08 \ . \end{aligned} \tag{SB.6}$$

The physical length of the telescope is given by $L = f_1 + f_2$. If we use this, eliminate f_2 from Eq. (SB.6), and use the given values for $2w_0'$ and w_0, we finally obtain

$$\begin{aligned} f_1 &= \frac{L}{1 - 1/\Gamma} = -24 \text{ mm} \ , \\ f_2 &= L - f_1 = 74 \text{ mm} \ . \end{aligned}$$

3. The change of intensity along the propagation direction of the Gauss beam can be derived from Eqs. (A85) and (A86) to

$$I(z) = \frac{I_0}{1 + (z - z_\mathrm{w})^2 / z_\mathrm{R}^2} \ ,$$

where z_w is the position of the beam waist and z_R the Rayleigh length. The given definition of the depth of focus simply means that

$$I(z = z_\mathrm{w} + \frac{\Delta z}{2}) = 0.8 \cdot I_0 \ . \qquad \text{(SB.7)}$$

From Eq. (SB.7) follows that

$$\begin{aligned} 0.8 \cdot I_0 &= \frac{I_0}{1 + \left(\frac{\Delta z/2}{z_\mathrm{R}}\right)^2} \\ \Rightarrow \frac{5}{4} &= 1 + \left(\frac{\Delta z/2}{z_\mathrm{R}}\right)^2 \ . \end{aligned}$$

With $\Delta z = 1\,\text{mm}$, we obtain $z_\mathrm{R} = 1\,\text{mm}$. Using the definition of the Rayleigh length in Eq. (A83) the focus diameter then yields for $\lambda = 632.8\,\text{nm}$

$$2w_0 = 2\sqrt{\frac{\lambda z_\mathrm{R}}{\pi}} = 0.0284 \text{ mm} \approx 28 \ \mu\text{m} \ .$$

The divergence angle is calculated from Eq. (A87) via

$$\varepsilon = \frac{\lambda}{\pi w_0} = \frac{w_0}{z_\mathrm{R}} = 0.01419 \text{ rad} \ .$$

With the given beam diameter $2w_0 = 8$ mm in front of the focusing lens, we find a focal length of

$$\begin{aligned} \varepsilon &= \frac{2w_0}{2f} \\ \Rightarrow f &= \frac{w_0}{\varepsilon} = 282 \text{ mm} \ . \end{aligned}$$

4. The beam propagation behind the telescope is shown in Figure SB.2. For the calculation of the beam parameters behind the lens, we use the formalism for the complex beam parameter q in Eq. (A81). It is obvious that q is purely imaginary

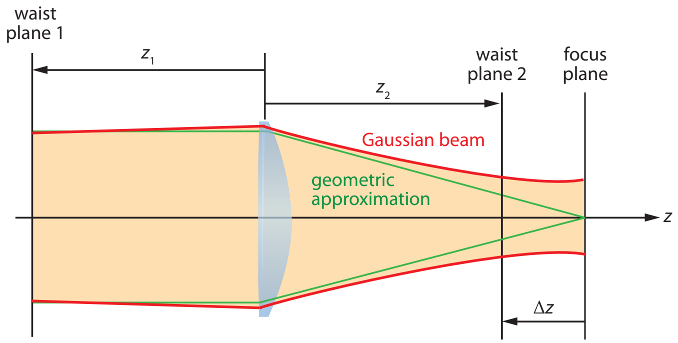


Figure SB.2 Gauss beam transformation through a telescope systems.

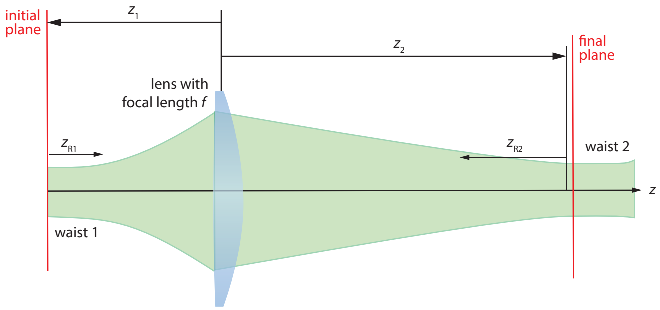


Figure SB.3 Gauss beam transformation by a focusing lens.

in the waist plane 1 (i.e. $r_C = \infty$). The same also holds for waist plane 2. From Eq. (A89) follows that

$$q_2 = \frac{Aq_1 + B}{Cq_1 + D}$$

with

$$q_1 = i \cdot z_{R1} = i \left(\frac{\pi \cdot w_{01}^2}{\lambda} \right) \quad .$$

The matrix elements can then be calculated with Figure SB.3 via

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 - z_2/f & (1 - z_2/f) \cdot z_1 + z_2 \\ -1/f & -z_1/f + 1 \end{pmatrix} \quad . \tag{SB.8}$$

From the boundary condition that q_2 is purely imaginary, we obtain the condition

$$q_2 = \frac{(BD + AC \cdot z_{R1}^2) + i(AD - BC)}{D^2 + C^2 \cdot z_{R1}^2}$$

from which directly derive

$$BD + AC \cdot z_{R1}^2 \equiv 0 \ . \qquad \text{(SB.9)}$$

This is identical with a condition for z_2, as this is the only variable element in the ABCD matrix in Eq. (SB.8). Solving Eq. (SB.9) for z_2 while using the matrix elements from Eq. (SB.8), we find

$$z_2 = f \cdot \frac{z_1^2 - z_1 \cdot f + z_{R1}^2}{(z_1 - f)^2 + z_{R1}^2} \ . \qquad \text{(SB.10)}$$

Inserting the values $z_1 = 300 \text{ mm}, f = 282 \text{ mm}, \lambda = 632.8 \text{ nm}, \text{ and } w_{01} = D/2 = 4 \text{ mm}$ leads to

$$\begin{aligned} z_{R1} &= \frac{\pi w_{01}^2}{\lambda} = 79.43 \text{ m} \ , \\ z_2 &= 282 \text{ mm} \ . \end{aligned}$$

For the waist position relative to the image-side focal point of the lens, we obtain

$$\Delta z_2 = f - z_2 = -0.23 \ \mu\text{m} \ .$$

The waist thus lies only 0.23 µm behind the geometric focus; actually opposite to what is shown in Figure SB.2. This is easy to understand, as the beam has a certain divergence going into the focusing lens which needs to be compensated for. In the case of a vanishing divergence (i.e., very strongly expanded beams), the location of the geometric focus and the beam waist are identical.

5. We start again from Eq. (A89), that is,

$$q_2 = \frac{Aq_1 + B}{Cq_1 + D}$$

with the ABCD matrix for the telescope

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} -f_2/f_1 & f_1 + f_2 \\ 0 & -f_1/f_2 \end{pmatrix} = \begin{pmatrix} 1/\Gamma & L \\ 0 & \Gamma \end{pmatrix}$$

in which $L = f_1 + f_2$ is again the length of the telescope. For the object-side waist at $z = 0$ directly in front of the telescope, at which q is purely imaginary, we have

$$q = z - iz_R = -iz_R \ .$$

Behind the telescope, we find for the complex beam parameter

$$\begin{aligned} q_2' &= z' - iz_R' \\ &= \frac{-iz_R/\Gamma + L}{-iz_R \cdot 0 + \Gamma} \\ &= \frac{L}{\Gamma} - \frac{iz_R}{\Gamma^2} \ . \end{aligned}$$

If we compare the imaginary parts, we see that

$$z'_{\mathrm{R}} = \frac{z_{\mathrm{R}}}{\Gamma^2} \ .$$

Using Eqs. (A83) and (A87)

$$\begin{aligned} z_{\mathrm{R}} &= \frac{\pi w_0^2}{\lambda} \ , \\ \varepsilon &= \frac{w_0}{z_{\mathrm{R}}} \ , \\ w'_0 &= \frac{w_0}{|\Gamma|} \ , \\ \varepsilon' &= \varepsilon \cdot |\Gamma| \end{aligned}$$

it follows that

$$z'_{\mathrm{R}} = \frac{w'_0}{\varepsilon'} = \frac{1}{\Gamma^2} \cdot \frac{w_0}{\varepsilon}$$

and

$$w'_0 \cdot \varepsilon' = w_0 \cdot \varepsilon \ .$$

Thus, the Rayleigh lengths transform with the inverse square of the telescope's reciprocal value of the angular magnification. Moreover, the product of beam waist and divergence angle is a constant. This is equivalent to the M factor in Eq. (B37). As a rule to remember: If the waist is magnified/de-magnified by a factor of $|\Gamma|$, the divergence is de-magnified/magnified by the same factor $|\Gamma|$.

PB.4
Laser power

Calculate the cw power of an Nd:YAG laser as a function of the degree of reflectance of the output mirror for various pumping power values of 1 kW, 2 kW, and 4 kW. Determine the optimal decoupling degree $T = 1 - R$. Use MathCAD or a similar program to calculate the profile of the output power for various output mirrors first and then attempt to find an analytical solution for the optimal degree of out-coupling (i.e., maximum output power). Use the following values:

- Saturation intensity: $I_{\mathrm{s}} = 2.2$ kW/cm^2.
- Laser rod dimensions: 0.5 cm (diameter), 10 cm (length).
- Transmittance in resonator: $T_{\mathrm{i}} = 0.95$.
- Pump efficiency $\eta_{\mathrm{pump}} = 5.5\%$.

Solution:

The formula describing the output power is given by (see also Eq. (B.28))

$$P_{\mathrm{out}} = \left(\frac{\mathrm{R}-1}{\mathrm{R}+1}\right) \cdot \left(\frac{\eta \text{ pump}}{\ln(T_{\mathrm{i}}\sqrt{\mathrm{R}})} P_{\mathrm{pump}} + \frac{V_{\mathrm{g}}}{L_{\mathrm{g}}} I_{\mathrm{s}}\right) \ , \qquad \text{(SB.11)}$$

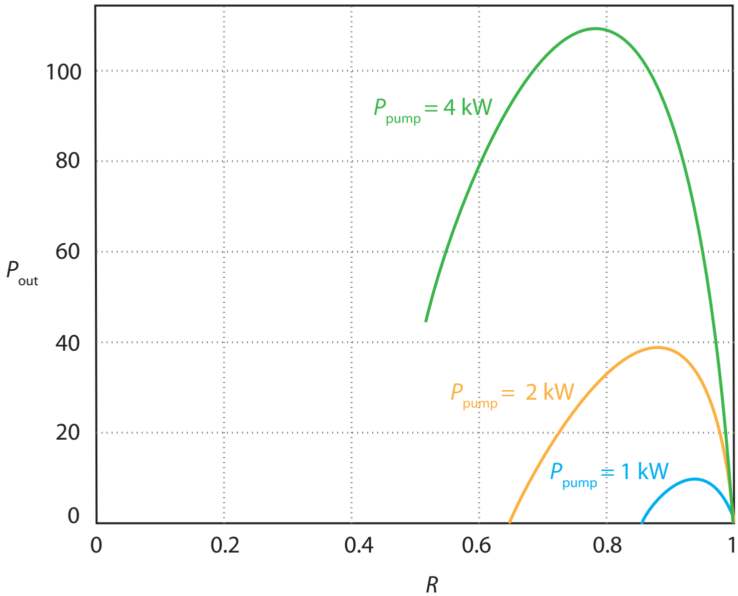


Figure SB.4: Output cw power for various pump powers as a function of the outcoupler reflectance.

with the reflectance of the output mirror R, the pumping power P_{pump}, the saturation intensity I_{s}, and the volume of the gain material $V_{\mathrm{g}} = \frac{\pi}{4} d^2 L_{\mathrm{g}}$. Figure SB.4 depicts the output curves obtained from a numerical calculation Eq. (SB.7) for three different pumping power values $P_{\mathrm{pump}} = 1\ \mathrm{kW}, 2\ \mathrm{kW}, 4\ \mathrm{kW}$. The exact analytical determination of the peak of the function via the derivative $\mathrm{d}P_{\mathrm{out}}/\mathrm{d}R = 0$ is impossible, since R appears directly and in the argument of the logarithm. The conditional equation then becomes transcendental and can therefore not be solved explicitly. The literature contains various approaches for approximate solutions[1].
If, for approximation, the value of R_2 was taken to be close to 1, a Taylor series could be developed for small values $x = 1 - R$. Using the following approximation is particularly elegant

$$\ln(R) \approx \frac{2(R-1)}{1+R} \ . \qquad \text{(SB.12)}$$

For values R close to 1, Figure SB.5 shows the relative error $\Delta f/f$ of this approximation. It is evident that the error is lower than 1% up to $R > 0.7$. With this approximation, we can rewrite Eq. (SB.11) in the following manner:

$$P_{\mathrm{out}} = \frac{0.5\ln(R)\cdot\eta_{\mathrm{pump}}}{\ln(T_{\mathrm{i}}) + 0.5\ln(R)}\left[P_{\mathrm{pump}} + P_{\mathrm{s}}\cdot(\ln(T_{\mathrm{i}}) + 0.5\ln(R))\right] \ ,$$

1) see e.g. Rigrod, W.W. (1978) *IEEE J. Quantum Electron.*, **14**, 377 or Schindler, G. M. (1980) *IEEE J. Quantum Electron.*, **16**, 546.

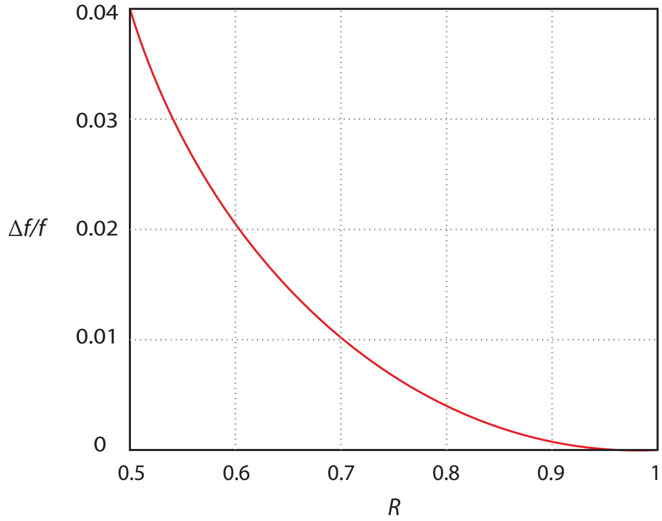


Figure SB.5: Relative error of the approximation in Eq. (SB.12).

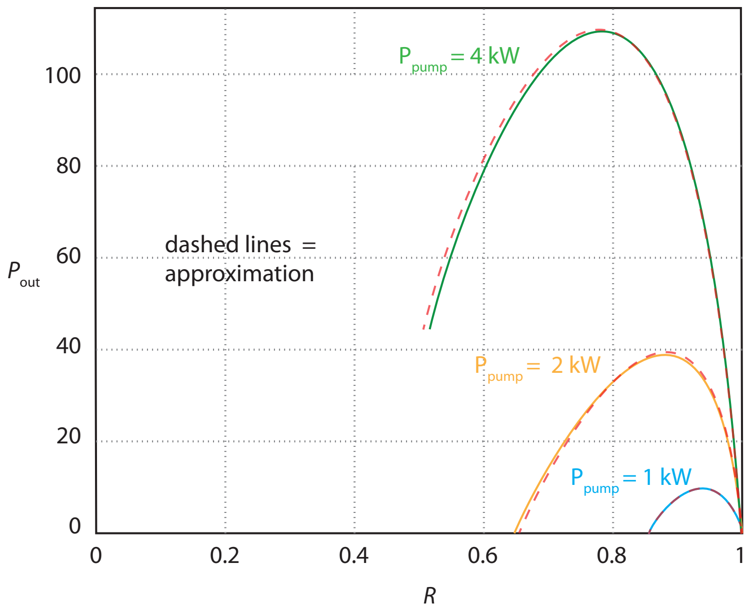


Figure SB.6 Output cw powers for various pump powers as a function of the outcoupler reflectance. Comparison of exact solution (solid line) and approximation (dashed line).

Table SB.1 Optimum degrees of out-coupling for various pumping conditions.

P_{pump} (kW)	R	T_{i}	P_{out} (W)	Efficiency η
1000	0.943	0.057	7.3	0.7%
2000	0.882	0.118	33.4	1.7%
4000	0.802	0.198	102.5	2.6 %

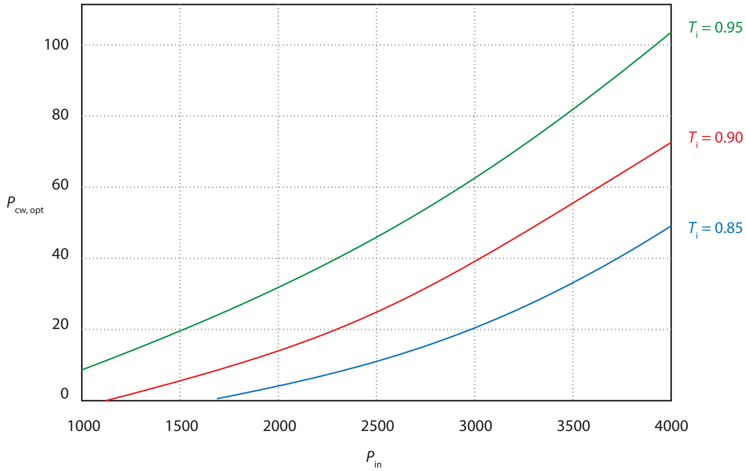


Figure SB.7 Maximum cw power for various internal losses as a function of the pumping power.

in which we have used the saturation power $P_{\text{s}} = I_{\text{s}} \cdot \frac{\pi}{4}\, d^2$. With the substitution $y = \ln(R)$, we obtain

$$P_{\text{out}} = 0.5 \cdot \eta_{\text{pump}} \cdot y \cdot \frac{[P_{\text{pump}} + P_{\text{s}} \cdot (\ln(T_{\text{i}}) + y/2)]}{\ln(T_{\text{i}}) + y/2} \quad .$$

Figure SB.6 gives an indication of the good quality of the approximation. The dashed lines indicate the approximations, while the continuous lines correspond to the analytically exact solutions. If one takes the approximate curve and forms the derivative with respect to y and sets it equal to zero, then the maximum can be derived from the relatively simple quadratic equation

$$y_{\max} = -2\ln(T_{\text{i}}) - 2\sqrt{\frac{P_{\text{pump}}}{P_{\text{s}}}\ln(T_{\text{i}})} \quad .$$

By using $R = \mathrm{e}^y$, we obtain the approximate values for the optical degree of "outcoupling" shown in Table SB.1. Figure SB.7 shows the maximum cw power for a given saturation power and various internal losses as a function of the pumping power.

PB.5
Nd:YAG laser

For photocoagulation, a frequency-doubled cw Nd:YAG laser is used. The Nd:YAG resonator consists of a concave mirror with a radius of curvature of 250 mm and a flat decoupling mirror. What is the maximum distance $L_{\max}$ between the mirrors to obtain a stable configuration? Is it possible to design a stable resonator made of two convex mirrors with the same radius of curvature and same distance $L_{\max}$?

Solution:

In Eqs. (B.34) and (B.35), we have

$$g_1 = 1 + \frac{L_\mathrm{R}}{r_1} \quad ,$$
$$g_2 = 1 + \frac{L_\mathrm{R}}{r_2} \quad .$$

In our case, $L_\mathrm{R} = L_{\max}$, $r_1 = -250\,\mathrm{mm}$, and $r_2 = \infty$. Hence, we can check the stability condition (B.36) given by $0 \leq g_1, g_2 \leq 1$ and find

$$0 \leq \left(1 + \frac{L_{\max}}{-250\,\mathrm{mm}}\right) \leq 1$$
$$\Rightarrow 0 \leq L_{\max} \leq 250\,\mathrm{mm}$$

Thus, the maximum length of the resonator is $L_{\max} = 250\,\mathrm{mm}$. In this case, we have a hemispherical resonator (Figure B.11).
In the case of two identical convex mirrors, we have $r_1 = r_2 = 250\,\mathrm{mm}$ and thus stability if

$$0 \leq \left(1 + \frac{L_{\max}}{r_1}\right) \cdot \left(1 + \frac{L_{\max}}{r_1}\right) \leq 1$$
$$\Rightarrow r_1 \leq -\frac{L_{\max}}{2} \quad .$$

In the stability diagram in Figure SB.8, this corresponds to an area outside of the regions of stability. Accordingly, *no stable* resonators exist with convex mirrors only, provided the resonator is empty as it was assumed in this approach. The situation may be different in the case of resonators filled with gain material.

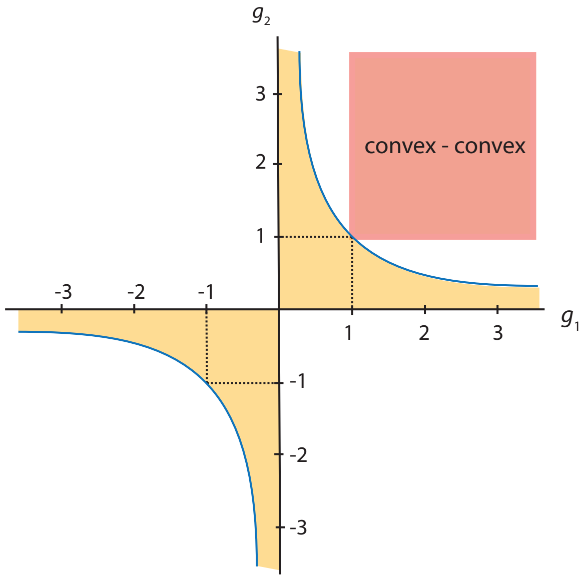


Figure SB.8 Stability diagram for convex-convex laser resonators.

PB.6
Thermal lens

An active laser medium within a laser resonator usually generates a so-called *thermal lens* due to thermal effects. Consider whether this will be a convergent or a divergent lens. For this purpose, look up the temperature dependence of the refractive index of laser materials on the internet! For building a model, let us consider a confocal-planar resonator of length L_R. For simplification, let the thermal lens with focal length f (sign!) be positioned exactly in the middle. The planar mirror is the output mirror. Calculate the stability condition as a function of the focal length of the thermal lens. How is the decoupled Gauss bundle changed by the lens (waist, divergence)?

Solution:

The temperature dependence of the refractive index of crystalline Nd:YAG is $dn/dT = +9.86 \times 10^{-6}\ \mathrm{K}^{-1} > 0$, that is, the refractive index increases with increasing temperature[2]. Heat is introduced to the resonator by the strong radiation field and the pumped light absorbed in the rod. Since the cooling of solid-state lasers is provided from the periphery, the rod (laser material) is hotter on the inside. This

2) Please refer to http://wr.lib.tsinghua.edu.cn/sites/default/files/1101116431461.pdf.

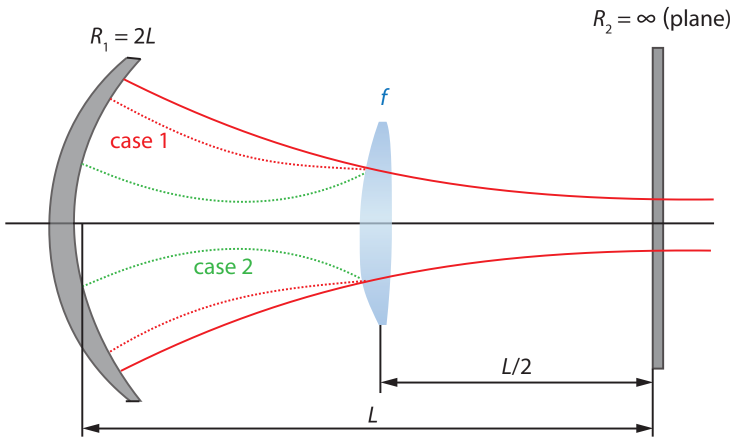


Figure SB.9 Geometry for thermal lens inside a resonator.

applies to longitudinal laser pumping as well. Accordingly, we have

$$T(r=0) > T(r_{\max}) \text{ and/or}$$
$$n(r=0) > n(r_{\max}).$$

In a first approximation, the refractive index profile can be assumed to be quadratic, that is,

$$n(r) = n_0 - ar^2 \ .$$

This corresponds to a radial gradient lens with positive refractive power. The thermally heated rod thus acts as a convergent lens in the resonator with $f > 0$.
A planar-confocal resonator with decoupling at the plane mirror is parametrized by $g_1 = 1/2$ and $g_2 = 1$. This means that the concave mirror has a radius of $R_1 = 2L$, (Figure SB.9). Applying Eq. (A86) and using the parameters for resonator stability, the waist radius w_0 on the plane output mirror is given by (neglecting any thermal lens effects)

$$w_2 = w_0 = \sqrt{\frac{\lambda L}{\pi} \cdot \sqrt{\frac{g_1}{g_2 \cdot (1 - g_1 g_2)}}} = \sqrt{\frac{\lambda L}{\pi}}$$

from which the divergence of the laser beam follows as

$$\varepsilon = \frac{\lambda}{\pi w_0} = \sqrt{\frac{\lambda}{\pi L}} \ .$$

Here, we have assumed that the output mirror has no refractive power.
We assume the thermal lens to have a refractive power of $\mathcal{D} = 1/f$ and to be located in the middle of the resonator. The ABCD matrix (internal contour matrix) of the

resonator (for a start upstream of mirror 1) is given by

$$\underline{\mathbf{M}} = \begin{pmatrix} 1 & 0 \\ -\frac{2}{R_1} & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & \frac{L}{2} \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -\mathcal{D} & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & \frac{L}{2} \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & \frac{L}{2} \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -\mathcal{D} & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & \frac{L}{2} \\ 0 & 1 \end{pmatrix} \ .$$

With $R_1 = 2L$, this becomes

$$\underline{\mathbf{M}} = \begin{pmatrix} 1 - 2L\mathcal{D} + \frac{1}{2}L^2\mathcal{D}^2 & 2L - \frac{3}{2}L^2\mathcal{D} + \frac{1}{4}L^3\mathcal{D}^2 \\ -\frac{1}{L} + \frac{1}{2}L\mathcal{D}^2 & -1 - \frac{1}{2}L\mathcal{D} + \frac{1}{4}L^2\mathcal{D}^2 \end{pmatrix} \ . \tag{SB.13}$$

The general stability condition

$$\frac{|A + D|}{2} \leq 1$$

leads to the following condition using the matrix elements of Eq. (SB.13):

$$\left| -\frac{5}{2}L\mathcal{D} + \frac{3}{4}L^2\mathcal{D}^2 \right| \leq 2 \ .$$

Resolving this quadratic inequality while taking into consideration that L and $\mathcal{D}$ are positive leads to the two solutions

$$\begin{aligned} &\text{Case 1}: \ f \geq \frac{3L}{4} \\ &\text{Case 2}: \ \frac{L}{4} \leq f \leq \frac{L}{2} \ . \end{aligned}$$

Case 1 describes the situation of an increasing thermal lens at higher core temperature. A threshold exists at which the resonator obviously becomes unstable. This occurs if the focal length of the lens is smaller than $3/4$ of the resonator length. In addition to case 1 of a relatively weak thermal lens effect with large f, a second solution of finite size exists for a larger thermal lens (case 2). This range has an intermediate waist. Both solutions are shown by dashed lines in the Figure SB.11. The solution areas are presented in Figure SB.10a as an f/L diagram and in Figure SB.10b as a stability condition.
Using Eq. (A89) with the ABCD values from Eq. (SB.13), the waist radius at the output mirror in the presence of a thermal lens with refractive power $\mathcal{D}$ is given by

$$\begin{aligned} w_2^2 &= \frac{2\lambda|B|}{\pi \cdot (A+1)^2} \cdot \sqrt{\frac{2 + A + D}{2 - (A + D)}} \\ &= \frac{2L\lambda}{\pi} \cdot \frac{|2 - \frac{3}{2}L\mathcal{D} + \frac{1}{4}L^2\mathcal{D}^2|}{(2 - 2L\mathcal{D} + \frac{1}{2}L^2\mathcal{D}^2)^2} \cdot \sqrt{\frac{2 - \frac{5}{2}L\mathcal{D} + \frac{3}{4}L^2\mathcal{D}^2}{2 + \frac{5}{2}L\mathcal{D} - \frac{3}{4}L^2\mathcal{D}^2}} \end{aligned}$$

Figure SB.11 shows the waist radius versus the functional length (Figure SB.11a) and versus the refractive power (Figure SB.11b).

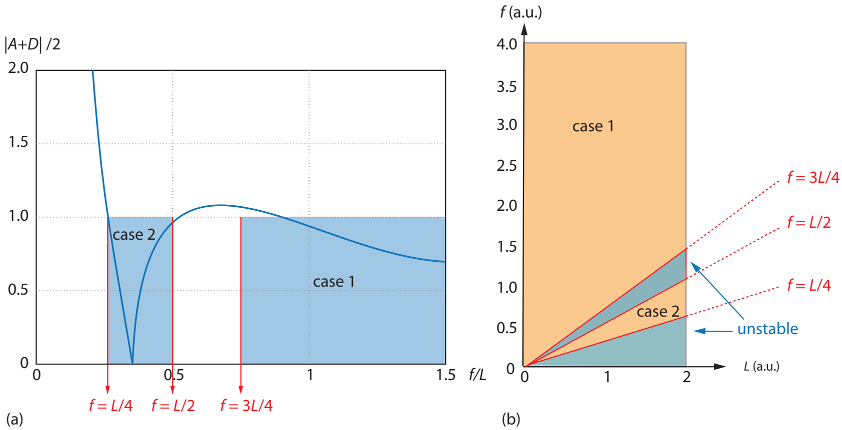


Figure SB.10 Stability condition and areas for a resonator with length L and in the prescence of a thermal lens with focal length f.

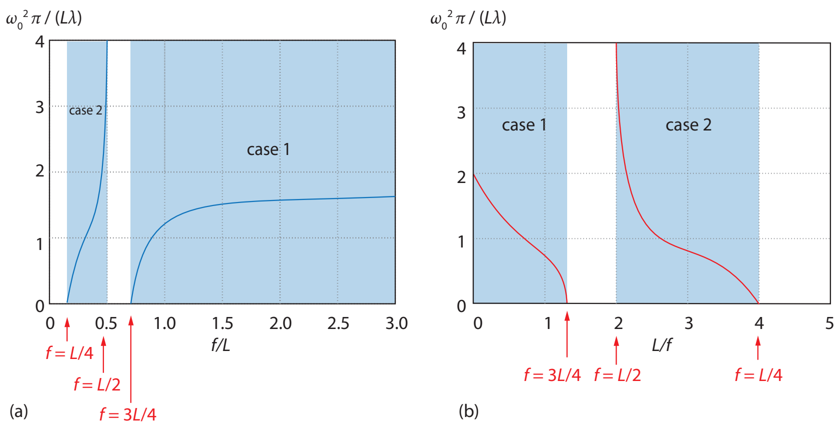


Figure SB.11 Waist radius at output coupler as a function of (a) the focal length and (b) the refractive power of the thermal lens.

PB.7
Q-switch

Consider a Q-switched Nd:YAG laser with a resonator/gain medium length of $L_\mathrm{g} = 30$ cm. The rod-shaped gain medium has a diameter of $d = 1$ cm and a gain cross-section of $\sigma = 3 \times 10^{-19}/\mathrm{cm}^{-3}$. It is actively used at $\approx 50\%$. Estimate the peak power, pulse duration, and pulse energy. We start with the following assumptions:

1. The number of excited atoms n_i is $4\times$ the threshold inversion multiplied by the active volume.
2. The resonator quality is switched by changing the transmission from 0% - 100% and by using mirrors with a reflectance of $R = \sqrt{R_1 R_2} = 0.5$. The peak power is approximately given by

$$P_\mathrm{peak} \approx \frac{\Delta N_\mathrm{i} \hbar\omega}{2\tau_\mathrm{res}} \quad , \tag{B45}$$

where the photons' lifetime in the resonator is $\tau_\mathrm{res} = L/(c - cR)$ and c is the speed of light. ΔN_i is the initial inversion before switching and the switching time is taken to be very short. The equation can be understood as a process at which the laser level is completely emptied within the resonator attenuation time. Note that losses due to decoupling are predominant. The pulse duration is approximately $3\tau_\mathrm{res}$.

Solution:

The threshold inversion density is given by

$$L_\mathrm{g} N_\mathrm{th} = -\frac{1}{2} \cdot \frac{\ln R}{\sigma} \quad .$$

The cross-selectional area of a 50%-filled circular cross-section of a Nd:YAG rod with a diameter of $d = 1$ cm is

$$A = \frac{1}{2} \cdot \frac{\pi}{4} d^2 = 0.393 \ \mathrm{cm}^2 \quad .$$

The active volume then becomes

$$V = A \cdot L_\mathrm{g} = 3.93 \ \mathrm{cm}^3 \quad .$$

With a gain cross-section of $\sigma = 3 \times 10^{-19}$ cm^2, the inversion at the threshold follows to

$$N_\mathrm{th} = -\frac{\ln R}{2L_\mathrm{g} \cdot \sigma} = 1.16 \times 10^{17} \ \mathrm{cm}^{-3} \quad .$$

Thus, the number of excited atoms is

$$n_\mathrm{i} = 4 N_\mathrm{th} \cdot V = 1.82 \times 10^{18} \quad .$$

The round-trip time of the resonator with a length of $L = 30\,\mathrm{cm}$ and a reflectance R is determined by

$$\tau_{\mathrm{res}} = \frac{L}{c(1-R)} = 2.0\ \mathrm{ns}\ \ .$$

The pulse duration is then approximately

$$\tau = 3\tau_{\mathrm{res}} = 6.0\ \mathrm{ns}\ \ .$$

Accordingly, with Planck's constant $h = 6.63\times 10^{-34}$ Js, the peak power is

$$P_{\mathrm{peak}} = \frac{n_{\mathrm{i}} \cdot h\nu}{2\tau_{\mathrm{res}}} = 8.51\times 10^{7}\ \mathrm{W}\ \ .$$

Assuming the pulse to be triangular in shape and $\overline{P} = 0.5\ P_{\mathrm{peak}}$, the pulse energy follows as

$$E_{\mathrm{p}} = \frac{1}{2}\tau \cdot P_{\mathrm{peak}} = 0.26\ \mathrm{J}\ \ .$$

According to literature[3)], the energy density bulk damage threshold of a Nd:YAG crystal as a function of the pulse duration is approximately given by

$$w_{\mathrm{dam}} = 50\frac{\mathrm{J}}{\mathrm{cm}^2}\sqrt{\frac{\tau}{1\ \mathrm{ns}}}\ \ .$$

For a pulse duration of 6 ns, the energy is then

$$E_{\mathrm{dam}} = A \cdot w_{\mathrm{dam}} = 50\frac{\mathrm{J}}{\mathrm{cm}^2}\sqrt{6} \cdot 0.393\ \mathrm{cm}^2 = 48\ \mathrm{J}\ \ .$$

The damage threshold is thus not a critical factor in the case considered here.
The damage thresholds of the coatings are, however, more critical. The website www.lasermaterials.com can be used to find a threshold of $1.4\ \mathrm{GW} \cdot \mathrm{cm}^{-2}$ for anti-reflection coatings and pulses of less than $\tau < 20\,\mathrm{ns}$. In the present case, the intensity above which the coating is damaged is

$$I_{\mathrm{dam}} = \frac{P_{\mathrm{peak}}}{A} = 0.22\ \mathrm{GW} \cdot \mathrm{cm}^{-2}\ \ .$$

This means that the layers should tolerate this radiation exposure as well.

3) Do, B.T. and Smith, A.V. (2009) *Appl. Opt.*, **48**, 3509.

PB.8
Ultra-short light pulses and self-phase modulation

The optical Kerr effect (Section 9.5), that is, the dependence of the refractive index on the intensity, is a third-order non linear effect. It is described by $n = n_0 + n_2 I$, where the nonlinear refractive index n_2 has the following values:

- Glass (BK7): $n_2 = 5 \times 10^{-15}\,\mathrm{cm^2/W}$
- Water: $n_2 = 10^{-16}\,\mathrm{cm^2/W}$
- Doped fiber: $n_2 = 10^{-10}\,\mathrm{cm^2/W}$

1. Show that a thin plate of BK7 with a thickness of 5 mm has the effect of a lens with a refractive index of n_l in the case of a 100 fs pulse at a wavelength of approximately 550 nm and approximately plane wave front (Gaussian mode, waist radius 0.5 mm). Calculate the focal length as a function of realistic pulse energies (in the range from 1 nJ - 10 µJ). Make use of the fact that the effect of a lens on a plane wave can be described by the phase term
$$\exp\left(\frac{ik(x^2+y^2)}{2n_\mathrm{l}}\right) \quad . \tag{B46}$$
2. The phase of the pulse also changes during passage through the medium according to
$$\varphi(t) = -2\pi\,\frac{L}{\lambda}\,n_2\,I(t) \quad . \tag{B47}$$
Compare a Gaussian pulse and a sech^2 pulse with a pulse duration of τ_0 for which the time dependence of the field strength amplitudes shall be given by $\exp(-t^2/\tau_0^2)$ and $\mathrm{sech}(t/\tau_0)$, respectively. Calculate the frequency response $\omega(t) = \mathrm{d}\varphi(t)/\mathrm{d}t$ for both pulse forms after passage through 10 mm of glass. Neglect self-focusing and group velocity dispersion. Assume the bundle to be approximately collimated when it travels through the material. Do we have an up- or a down-chirp?
3. Calculate the spectrum of the Gaussian pulse upstream and downstream of the material including the full widths at half maximum of the spectra. Regarding the spectra after passage through the material, let us consider the chirp effect to be a small correction that can be approximated to simplify the calculation. Does the product of pulse duration and spectral width give you any hints?

Solution:

1. The spatial distribution of the Gaussian bundle is given by
$$I(r) = I_\mathrm{peak} \cdot \exp\left[-2\left(\frac{r}{w_0}\right)^2\right] \quad .$$

From this follows the beam power by means of integration

$$P = \int_0^\infty I(r) 2\pi \, \mathrm{d}r = \frac{\pi w_0^2}{2} I_{\text{peak}} \ .$$

As a consequence, we obtain the relation to the pulse energy E_p for a pulse duration of τ_0 given by

$$I_{\text{peak}} = \frac{2E_\text{p}}{\pi \tau_0 w_0^2} \ .$$

Considering the Kerr effect to be a small disturbance, the distribution of intensities can be approximated as being quadratically around the peak intensity in order to determine the refractive index profile. Hence, we have

$$\begin{aligned} I(r) &= I_{\text{peak}} \cdot \exp\left[-2\left(\frac{r}{w_0}\right)^2\right] \\ &\approx I_{\text{peak}} \cdot \left(1 - \frac{2r^2}{w_0^2}\right) = \frac{2E_\text{p}}{\pi \tau_0 w_0^2} \cdot \left(1 - \frac{2r^2}{w_0^2}\right) \ . \end{aligned}$$

The equation describing the Kerr effect $n = n_0 + n_2 \cdot I$ leads to

$$\begin{aligned} n(r) &= n_0 + \frac{2n_2 \cdot E_\text{p}}{\pi \tau_0 w_0^2} \cdot \left(1 - \frac{2r^2}{w_0^2}\right) \\ &= n_0 + \frac{2n_2 \cdot E_\text{p}}{\pi \tau_0 w_0^2} - \frac{4n_2 \cdot E_\text{p}}{\pi \tau_0 w_0^4} \cdot r^2 \ . \end{aligned} \tag{SB.14}$$

A quadratic gradient medium with a refractive index of (Table A.1)

$$n(r) = n_{\text{center}} \cdot \left(1 - \varepsilon^2 r^2\right) \tag{SB.15}$$

and with a length of L has a focal length of

$$f = \frac{1}{n_{\text{center}} \varepsilon \cdot \sin(\varepsilon L)} \ .$$

Using the approximation $\varepsilon \ll \pi/L$ (relatively small non-linearity) leads to

$$f \approx \frac{1}{n_{\text{center}} L \varepsilon^2} \ .$$

By comparing the coefficients of Eqs. (SB.14) and (SB.15), we deduce

$$\varepsilon^2 = \frac{8n_2 \cdot E_\text{p}}{\pi \tau_0 n_{\text{center}} w_0^4} \ .$$

With the approximation $n_{\text{center}} \approx n_0$ ($n_0 = 1.51872$ for BK7), the focal length can be written as

$$f = \frac{\pi \tau_0 \cdot w_0^4}{8 L n_2 E_\text{p}} \ .$$

By applying the given numbers, the focal length values in Table SB.2 can be calculated.

Table SB.2 Focal length values f for different pulse energies E_p.

E_p (nJ)	f (mm)
10,000	6.1
1000	61
100	610
10	6100
1	61,000

2. For a Gaussian pulse with an amplitude of

$$A_\mathrm{G}(t) = A_0 \cdot \exp\left(-\frac{t^2}{\tau_0^2}\right)$$

and an intensity of

$$I_\mathrm{G}(t) = I_\mathrm{peak} \cdot \exp\left(-\frac{2t^2}{\tau_0^2}\right) \quad ,$$

the phase as a function of time is given by

$$\varphi_\mathrm{G}(t) = -\frac{2\pi n_2 L I_\mathrm{peak}}{\lambda} \exp\left(-\frac{2t^2}{\tau_0^2}\right) \quad .$$

The corresponding frequency response is then determined by

$$\begin{aligned}\omega_\mathrm{G}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\varphi_\mathrm{G}(t) &= -2\pi\frac{L}{\lambda}n_2 \cdot \frac{\mathrm{d}}{\mathrm{d}t}\left[A_0^2 \cdot \exp\left(-\frac{2t^2}{\tau_0^2}\right)\right] \\ &= \frac{8\pi n_2 L I_\mathrm{peak}}{\lambda\tau_0^2} \cdot t \cdot \exp\left(-\frac{2t^2}{\tau_0^2}\right) \quad .\end{aligned}$$

In analogy, for the sech-shaped pulse, the pulse amplitude is given by

$$A_\mathrm{sech}(t) = A_0 \cdot \mathrm{sech}\left(\frac{t}{\tau_0}\right)$$

and the intensity by

$$I_\mathrm{sech}(t) = I_\mathrm{peak} \cdot \mathrm{sech}^2\left(\frac{t}{\tau_0}\right) \quad .$$

The phase as a function of time is then

$$\varphi_\mathrm{sech}(t) = -\frac{2\pi n_2 L I_\mathrm{peak}}{\lambda} \cdot \mathrm{sech}^2\left(\frac{t}{\tau_0}\right) \quad .$$

The corresponding frequency response is determined by

$$\omega_\mathrm{sech}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\varphi_\mathrm{sech}(t) = \frac{4\pi n_2 L I_\mathrm{peak}}{\lambda\tau_0} \cdot \frac{\sinh(t/\tau_0)}{\cosh^3(t/\tau_0)} \quad .$$

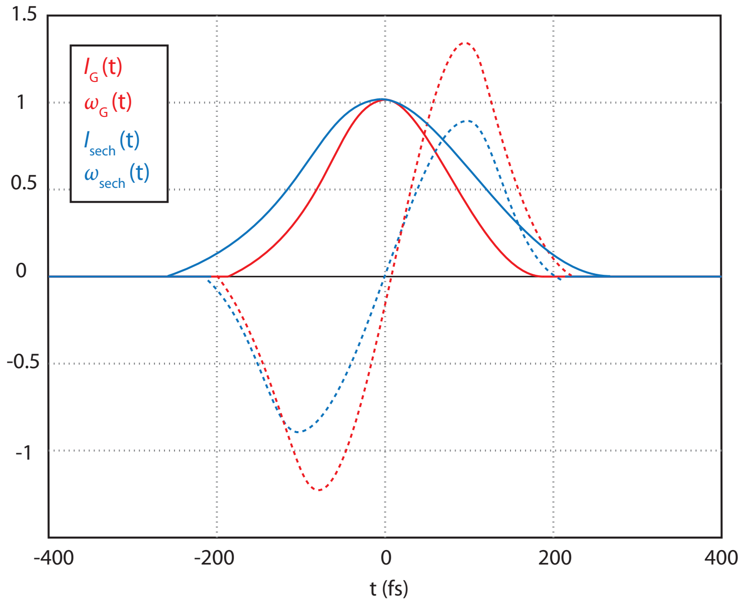


Figure SB.12 Pulse envelope and frequency chirp in a Gaussian and a sech^2 pulse in a nonlinear medium.

The two frequency profiles are shown together with the pulse envelopes in Figure SB.13. In both cases, for the main part of the pulse that contains energy fractions, $\mathrm{d}\omega/\mathrm{d}t > 0$. Thus, an up-chirp exists, whereby the Gaussian pulse (red) generates a larger chirp than the sech pulse (blue) due to its flank having a larger slope.

3. For a Gaussian pulse *prior to passage* through the medium, the amplitude is given by

$$A_{\mathrm{prior}}(t) = A_0 \cdot \exp\left(-\frac{t^2}{\tau_0^2}\right) \quad .$$

A Fourier transformation leads to

$$A_{\mathrm{prior}}(\nu) = \overline{A}_0 \cdot \exp\left(-\pi^2 \tau_0^2 \nu^2\right) \quad .$$

As a consequence, the power spectrum is determined by

$$S_{\mathrm{prior}}(\nu) = |A_{\mathrm{prior}}(\nu)|^2 = \overline{A}_0^2 \cdot \exp\left(-2\pi^2 \tau_0^2 \nu^2\right) \quad .$$

The full width at half maximum of the frequency spectrum is then

$$\Delta\nu_{1/2} = \frac{\sqrt{2\ln 2}}{\pi \cdot \tau_0} = \frac{0.375}{\tau_0}$$

so that the pulse duration-bandwidth product reads

$$\tau_0 \cdot \Delta\nu_{1/2} = 0.375 \ .$$

For a Gaussian pulse *after passage* through the medium, the amplitude is given by

$$A_{\text{after}}(t) = A_0 \cdot \exp\left(-\frac{t^2}{\tau_0^2} - i\omega(t)t\right) \ .$$

With a quadratic approximation of the frequency response in the exponent

$$\omega(t) = B \cdot t \cdot \exp\left(-\frac{2t^2}{\tau_0^2}\right) = Bt \cdot \left(1 - \frac{2t^2}{\tau_0^2}\right) \ ,$$

whereas we used the definition

$$B = \frac{8\pi n_2 L I_{\text{peak}}}{\lambda \tau_0^2} \ ,$$

we derive

$$\begin{aligned} A_{\text{after}}(t) &= A_0 \cdot \exp\left(-\frac{t^2}{\tau_0^2} - iBt^2 \cdot \left(1 - \frac{2t^2}{\tau_0^2}\right)\right) \\ &\approx A_0 \cdot \exp\left(-\frac{t^2}{\tau_0^2}\left(1 + iB\tau_0^2\right)\right) \ . \end{aligned}$$

After a Fourier transformation, the amplitude becomes

$$A_{\text{after}}(\nu) = \overline{A}_0^2 \cdot \exp\left(-\frac{\pi^2\tau_0^2\nu^2}{1 + iB^2\tau_0^4}\right) \ .$$

From this, the power spectrum follows to

$$S_{\text{after}}(\nu) = |A_{\text{after}}(\nu)|^2 = \overline{A}_0^2 \cdot \exp\left(-\frac{2\pi^2\tau_0^2\nu^2}{1 + B^2\tau_0^4}\right) \ .$$

The full width at half-maximum of the frequency spectrum is

$$\Delta\nu_{1/2} = \frac{\sqrt{2\ln 2}}{\pi\tau_0}\sqrt{1 + B^2\tau_0^4} = \frac{0.375}{\tau_0}\sqrt{1 + B^2\tau_0^4}$$

so that the pulse duration-bandwidth product results to

$$\tau_0 \cdot \Delta\nu_{1/2} = 0.375 \cdot \sqrt{1 + B^2\tau_0^4} \ .$$

The diagram in Figure SB.13a shows the intensity $I(t)$ and frequency $\omega(t, B)$ response for the finite value of the chirp with parameter B. The spectrum without (prior to transit through the medium) and with chirp (after transit) is presented in Figure SB.13b. The spectral broadening due to the chirp is clearly visible.

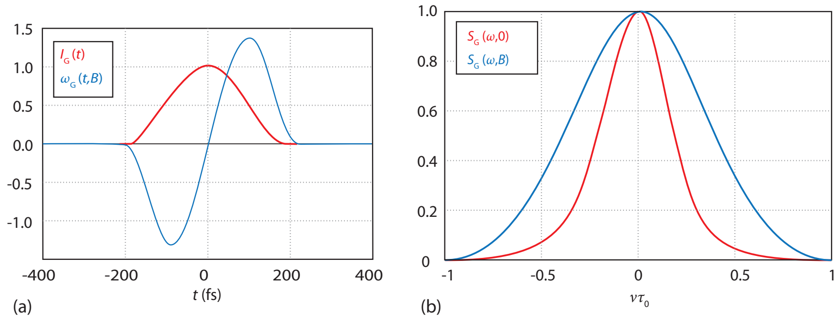


Figure SB.13 (a) Pulse envelope, frequency spectrum, and (b) power spectrum of a Gaussian pulse after passage through a nonlinear medium.

PB.9
Ultra-short light pulses in the presence of dispersion

Considering the propagation of ultra-short light pulses, it is necessary to take nonlinear effects and the effect of group velocity dispersion into account (Section 9.5). An ultra-short light pulse is given by

$$\psi(z,t) = \psi_0(z,t)\,\mathrm{e}^{2\pi i\nu_0 t - ikz} \quad . \tag{B48}$$

In space-time domain, ν_0 is the center frequency of the pulse and k the propagation constant determined by

$$k(\nu) = \frac{2\pi\nu\, n(\nu)}{c_0} \tag{B49}$$

with the speed of light in vacuum c_0 and the refractive index $n(\nu)$. In the so-called *SVE approximation* (slowly varying envelope), the propagation equation can be written as

$$\frac{\partial\psi_0}{\partial t^2} + \frac{4\pi i}{D_\nu}\frac{\partial\psi_0}{\partial z} = 0 \quad , \tag{B50}$$

where the dispersion coefficient is given by

$$D_\nu = \frac{1}{2\pi}\frac{\mathrm{d}^2 k}{\mathrm{d}\nu^2} = \frac{\mathrm{d}}{\mathrm{d}\nu}\left(\frac{1}{c_\mathrm{g}}\right) \tag{B51}$$

with the group velocity

$$\frac{1}{c_\mathrm{g}} = \frac{1}{2\pi}\frac{\mathrm{d}k}{\mathrm{d}\nu} \quad . \tag{B52}$$

The SVE approximation shows formal similarity to the paraxial wave equation (A79), which was used to derive the Gaussian modes. Accordingly, the solution for $\psi_0(z,t)$ will formally correspond to the solution for Eq. (A80) after appropriate substitution. We have

$$\psi_0(z,t) = A_0\sqrt{\frac{-iz_0}{z-iz_0}}\exp\left(\frac{i\pi\left(t-\frac{z}{c_\mathrm{g}}\right)^2}{D_\nu(z-iz_0)}\right) \quad . \tag{B53}$$

From this can be derived the pulse duration $\tau(z)$ and the so-called *chirp parameter* $a(z)$ which are given by

$$\tau(z) = \tau_0\sqrt{1+\left(\frac{z}{z_0}\right)^2} \tag{B54}$$

and $a(z) = z/z_0$ with the dispersion length $z_0 = \pi\tau_0^2/D_\nu$.

Table B.6 Dispersion parameters of different sorts of glass at a wavelength of 546 nm.

Material	**Refractive Index**	$\mathrm{d}n/\mathrm{d}\lambda$ (mm^{-1})	$\mathrm{d}^2n/\mathrm{d}\lambda^2$ (mm^{-2})
BK7	1.51872	−51.3514	269835
SF6	1.81265	−204.024	1305289
FK54	1.43815	−30.748	161347

Table SB.3 Comparison of variables to describe Gaussian bundles (space-angle domain) and Gaussian pulses (time-frequency domain).

Space-angle domain		**Time-frequency domain**	
Variable	Formula /sign	Variable	Formula /sign
Lateral space	x, r	Time	t
Beam width	w	Pulse length	τ
Wavelength	λ	Dispersion constant	D_ν
Angle of divergence	θ_0	Pulse broadening rate	$\frac{D_\nu}{\pi\tau}$
Rayleigh length	$z_0 = \frac{\pi w_0^2}{\lambda}$	Dispersion length	$z_0 = \frac{\pi\tau_0^2}{\mathcal{D}_\nu}$
Guoy phase	z/z_0	Chirp	$a = z/z_0$
Wavefront curvature	$\frac{1}{R} = \frac{z}{z^2+z_0^2}$	Chirp rate	$\varphi'' = \frac{2\pi}{D_\nu} \cdot \frac{z}{z^2+z_0^2}$

1. Describe the analogy between Gaussian pulses and Gaussian bundles. Which variables are equivalent? Which variables correspond to the bundle width, wavefront curvature, angle of divergence, and Rayleigh length?
2. Calculate the intensity of the pulse as a function of z.
3. Calculate the pulse width and chirp of an originally bandwidth-limited pulse of 100 fs at a wavelength of approximately 546 nm after passage through different BK7 glass rods with lengths of 5, 10, and 50 mm.
4. What happens to a pulse that propagates with a down-chirp into a medium with positive group velocity? Determine the optimal length of the glass block made of BK7 after which a pulse with a pulse duration of τ_1 and a negative chirp of a_1 has become a bandwidth-limited pulse. How short will the pulse have become by then?

Use the material parameters for a wavelength of 546 nm from Table B.6.

Solution:

1. In Table SB.3, the comparison of variables to describe Gaussian bundles and pulses is presented.

2. The amplitude is given by

$$\psi_0(z,t) = A_0 \cdot \sqrt{\frac{-iz_0}{z - iz_0}} \cdot \exp\left(i\frac{\pi}{D_\nu}\frac{(t - z/c_{\rm g})^2}{z - iz_0}\right) \quad .$$

From this follows the intensity

$$\begin{aligned} I(z,t) = |\psi_0(z,t)|^2 = I_0 \cdot \sqrt{\frac{z_0^2}{z^2 + z_0^2}} \cdot \exp\left(i\frac{\pi}{D_\nu}\frac{(t - z/c_{\rm g})^2}{z - iz_0} - i\frac{\pi}{D_\nu}\frac{(t - z/c_{\rm g})^2}{z + iz_0}\right) \\ = I_0 \cdot \frac{\tau_0^2}{\tau^2(z)} \cdot \exp\left(-2 \cdot \frac{(t - z/c_{\rm g})^2}{\tau^2(z)}\right) \quad . \end{aligned}$$

3. At first, we convert the dispersion coefficients from frequency to wavelength notation. To change of variables of the derivatives, we write

$$\begin{aligned} \lambda &= \frac{c}{\nu} \quad , \\ \frac{\mathrm{d}}{\mathrm{d}\nu} &= \frac{\mathrm{d}\lambda}{\mathrm{d}\nu} \cdot \frac{\mathrm{d}}{\mathrm{d}\lambda} = -\frac{c}{\nu^2} \cdot \frac{\mathrm{d}}{\mathrm{d}\lambda} = -\frac{\lambda^2}{c} \cdot \frac{\mathrm{d}}{\mathrm{d}\lambda} \quad , \\ \frac{\mathrm{d}^2}{\mathrm{d}\nu^2} &= -\frac{\lambda^2}{c} \cdot \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(-\frac{\lambda^2}{c} \cdot \frac{\mathrm{d}}{\mathrm{d}\lambda}\right) = -\frac{\lambda^2}{c} \cdot \left(-\frac{2\lambda}{c} \cdot \frac{\mathrm{d}}{\mathrm{d}\lambda} - \frac{\lambda^2}{c} \cdot \frac{\mathrm{d}^2}{\mathrm{d}\lambda^2}\right) \\ &= \frac{2\lambda^3}{c^2} \cdot \frac{\mathrm{d}}{\mathrm{d}\lambda} + \frac{\lambda^4}{c^2} \cdot \frac{\mathrm{d}^2}{\mathrm{d}\lambda^2} \quad . \end{aligned}$$

Applying the given values and using the data from Table B.6, we obtain

$$\begin{aligned} D_\nu = \frac{1}{2\pi} \cdot \frac{\mathrm{d}^2 k}{\mathrm{d}\nu^2} &= \frac{1}{2\pi} \cdot \frac{\mathrm{d}^2}{\mathrm{d}\nu^2}\left(\frac{2\pi}{c} \cdot \nu n\right) = \frac{1}{c}\frac{\mathrm{d}}{\mathrm{d}\nu}\left(n + \nu\frac{\mathrm{d}n}{\mathrm{d}\nu}\right) \\ &= \frac{1}{c}\left(2\frac{\mathrm{d}n}{\mathrm{d}\nu} + \nu\frac{\mathrm{d}^2 n}{\mathrm{d}\nu^2}\right) = \frac{\lambda^3}{c^2}\frac{\mathrm{d}^2 n}{\mathrm{d}\lambda^2} \\ &= 4.88 \times 10^{-18}\ \mathrm{s}^2\mathrm{mm}^{-1} \quad . \end{aligned}$$

The dispersion length is then given by

$$z_0 = \frac{\pi\tau_0^2}{D_\nu} = 64.4\ \mathrm{mm} \quad .$$

Accordingly, the pulse width and chirp values for $z = L$ are listed in Table SB.4. The non-linearity/Kerr effect influences the spectral shape of the beam and is a function of the pulse power/intensity. The dispersion affects the temporal form of the pulse and is a function of the pulse duration, but not of the intensity. Both effects increase with increasing path length in the medium.

4. A pulse with a down-chirp $\mathrm{d}\omega/\mathrm{d}t < 0$ travelling in a medium with a positive group velocity becomes phase-corrected with increasing propagation distance and thus approaches the ideal bandwidth-limited temporal form. The characteristic dispersion length for pulse duration τ_1 is

$$z_0 = \frac{\pi\tau_1^2}{D_\nu} \quad .$$

Table SB.4 Pulse width and chirp values for different BK7 rod lengths.

Variable	Formula	$z = 5\,\text{mm}$	$z = 10\,\text{mm}$	$z = 50\,\text{mm}$
Chirp	$a(z) = z/z_0$	0.078	0.155	0.777
Pulse width	$\tau(z) = \tau_0\sqrt{1 + (z/z_0)^2}$	100.30 fs	101.2 fs	126.6 fs

The chirp coefficient is given by

$$a(z) = z/z_0 \ .$$

If a_1 is the given chirp, for a travelled distance of

$$z = a_1 z_0 = \frac{\pi a_1 \tau_1^2}{D_\nu} \ ,$$

the present quadratic approximations yield just the level of compensation for the chirp to have disappeared. Here, the pulse duration is reduced to a value of

$$\tau_0 = \frac{\tau_1}{\sqrt{1 + a_1^2}} \ .$$